MTH 291 Practice Final Exam Key

1. $\frac{dy}{dx} = y - xy \quad y(0) = 2 \quad \Delta t = 0.1$

$n=0 \quad x_0 = 0 \quad y_0 = 2 \quad k_{01} = 0.1(2 - 0(2)) = 0.2$

$k_{02} = 0.1(2.1 - 0.05(2.1)) = .1995$

$k_{03} = 0.1(2.09975 - 0.05(2.09975)) = .19947625$

$k_{04} = 0.1(2.19947625 - 0.1(2.19947625)) = .1979528625$

$y_{n+1} = 2 + .1993175604 = 2.1993$

$n=1 \quad x_1 = 0.1 \quad y_1 = 2.1993 \quad k_{11} = 0.1(2.1993 - 0.1(2.1993)) = .197938\ldots$

$k_{12} = 0.1(2.098969 - 0.15(2.098969)) = .1784073..$

$k_{13} = 0.1(2.08921 - 0.15(2.08921)) = .1775825\ldots$

$k_{14} = 0.1(2.1775 - .2(2.1775)) = .1742066$

$y_n = 2.1993 + .180689 = 2.379989$

2. $y = x_1$

$y' = x_1' = x_2$

$y'' = x_2'$

$x_1' = x_2$

$x_2' = -x_1 + \cos 2t - 6\sin 2t$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos 2t - 6\sin 2t \end{pmatrix}$$

3. $\begin{pmatrix} 1-\lambda & 3 \\ -1 & -2-\lambda \end{pmatrix}$ $\quad (1-\lambda)(-2-\lambda) + 3 = 0 \quad \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\lambda^2 + \lambda - 2 + 3 = 0$

$\lambda^2 + \lambda + 1 = 0$

$$\begin{pmatrix} 1 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) & 3 \\ -1 & -2 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2}i & 3 \\ -1 & -\frac{3}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix}$$

$-x_1 = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)x_2 \quad \begin{pmatrix} \frac{3}{2} + \frac{\sqrt{3}}{2}i \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 + \sqrt{3}i \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 3 + \sqrt{3}i \\ -2 \end{pmatrix} e^{-\frac{1}{2}t}\left(\cos \frac{\sqrt{3}}{2}t + i \sin \frac{\sqrt{3}}{2}t\right) = e^{-\frac{1}{2}t}\begin{pmatrix} 3\cos \frac{\sqrt{3}}{2}t + 3i\sin \frac{\sqrt{3}}{2}t + \sqrt{3}i\cos \frac{\sqrt{3}}{2}t - \sqrt{3}\sin \frac{\sqrt{3}}{2}t \\ -2\cos \frac{\sqrt{3}}{2}t - 2i\sin \frac{\sqrt{3}}{2}t \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 3\cos \frac{\sqrt{3}}{2}t - \sqrt{3}\sin \frac{\sqrt{3}}{2}t \\ -2\cos \frac{\sqrt{3}}{2}t \end{pmatrix} e^{-\frac{1}{2}} + c_2 \begin{pmatrix} 3\sin \frac{\sqrt{3}}{2}t + \sqrt{3}\cos \frac{\sqrt{3}}{2}t \\ -2\sin \frac{\sqrt{3}}{2}t \end{pmatrix}$$

4. $\Psi = \begin{pmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{pmatrix}$

$\Psi' = \begin{pmatrix} -6e^{-2t} & e^{t} & 3e^{3t} \\ 4e^{-2t} & -e^{t} & -3e^{3t} \\ -4e^{-2t} & e^{t} & 0 \end{pmatrix}$

$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{pmatrix} =$

col 1

$\begin{pmatrix} -24e^{-2t} + 22e^{-2t} - 4e^{-2t} = -6e^{2t} \\ 18e^{-2t} - 18e^{-2t} + 4e^{-2t} = 4e^{-2t} \\ -18e^{-2t} + 12e^{-2t} + 2e^{-2t} = -4e^{-2t} \end{pmatrix}$

$\begin{pmatrix} -6e^{2t} & e^{t} & 3e^{3t} \\ 4e^{-2t} & -e^{t} & -3e^{3t} \\ -4e^{-2t} & e^{t} & 0 \end{pmatrix}$

col 2

$\begin{pmatrix} -8e^{t} + 11e^{t} - 2e^{t} = e^{t} \\ 6e^{t} - 9e^{t} + 2e^{t} = -e^{t} \\ -6e^{t} + 6e^{t} + e^{t} = e^{t} \end{pmatrix}$

they match so it is the fundamental solution

col 3

$\begin{pmatrix} -8e^{3t} + 11e^{3t} + 0 = 3e^{3t} \\ 6e^{3t} - 9e^{3t} + 0 = -3e^{3t} \\ -6e^{3t} + 6e^{3t} + 0 = 0 \end{pmatrix}$

5. $\vec{X}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\vec{X}$ $\quad \begin{pmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{pmatrix} \to (1-\lambda)(7-\lambda)+9 = 0$

$\lambda^2 - 8\lambda + 7 + 9 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda \pm 4$

$\begin{pmatrix} 1-4 & -3 \\ 3 & 7-4 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}$

$3x_1 + 3x_2 = 0$
$x_1 = -x_2$ $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -3 & -3 & 1 \\ 3 & 3 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -1/3 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 + x_2 = -\frac{1}{3}$ $\quad \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$

$X = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t} + c_2\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}e^{4t}\right]$

origin is repeller
$\lambda = 4$

6. $\Psi = \begin{pmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{pmatrix}$ $\qquad \Psi^{-1} = \frac{1}{7e^{-3t}}\begin{pmatrix} e^{3t} & 4e^{3t} \\ -2e^{-6t} & -e^{-6t} \end{pmatrix}$

$|\Psi| = -e^{-3t} + 8e^{-3t} = 7e^{-3t}$ $\qquad = \begin{pmatrix} \frac{1}{7}e^{6t} & \frac{4}{7}e^{6t} \\ -\frac{2}{7}e^{-3t} & -\frac{1}{7}e^{-3t} \end{pmatrix}$

$$\Psi^{-1}g = \begin{pmatrix} \frac{1}{7}e^{6t} & \frac{4}{7}e^{6t} \\ -\frac{2}{7}e^{-3t} & -\frac{1}{7}e^{-3t} \end{pmatrix}\begin{pmatrix} 3e^{t} \\ -t^2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7}e^{7t} - \frac{4}{7}t^2e^{6t} \\ -\frac{6}{7}e^{-2t} + \frac{1}{7}t^2e^{-3t} \end{pmatrix}$$

$$\int \Psi^{-1}g\,dt = \int\begin{pmatrix} \frac{3}{7}e^{7t} - \frac{4}{7}t^2e^{6t} \\ -\frac{6}{7}e^{-2t} + \frac{1}{7}t^2e^{-3t} \end{pmatrix}dt = \begin{pmatrix} \frac{3}{49}e^{7t} - \frac{2}{21}t^2e^{6t} + \frac{2}{63}te^{6t} - \frac{1}{189}e^{6t} \\ +\frac{3}{7}e^{-2t} - \frac{1}{21}t^2e^{-3t} - \frac{2}{63}te^{-3t} - \frac{2}{189}e^{-3t} \end{pmatrix}$$

$$\Psi\int\Psi^{-1}g\,dt = \begin{pmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{pmatrix}\begin{pmatrix} \frac{3}{49}e^{7t} - \frac{2}{21}t^2e^{6t} + \frac{2}{63}te^{6t} - \frac{1}{189}e^{6t} \\ \frac{3}{7}e^{-2t} - \frac{1}{21}t^2e^{-3t} - \frac{2}{63}te^{-3t} - \frac{2}{189}e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{49}e^{t} + \frac{2}{21}t^2 - \frac{2}{63}t + \frac{1}{189} - \frac{[illegible]}{7}e^{t} + \frac{4}{21}t^2 + \frac{8}{63}t + \frac{8}{189} \\ \frac{6}{49}e^{t} - \frac{4}{21}t^2 + \frac{4}{63}t + \frac{2}{189} + \frac{3}{7}e^{t} - \frac{1}{21}t^2 - \frac{2}{63}t - \frac{2}{189} \end{pmatrix} = \begin{pmatrix} -\frac{87}{49}e^{t} + \frac{2}{7}t^2 + \frac{2}{21}t + \frac{1}{21} \\ \frac{27}{49}e^{t} - \frac{5}{21}t^2 + \frac{2}{63}t \end{pmatrix}$$

7. $g = \begin{pmatrix} 3e^{t} \\ 0 \end{pmatrix}$ $\qquad h = \begin{pmatrix} 0 \\ -t^2 \end{pmatrix}$

Separately

$X_1(t) = \begin{pmatrix} Ae^{t} \\ Be^{t} \end{pmatrix}$ $\qquad X_1'(t) = \begin{pmatrix} Ae^{t} \\ Be^{t} \end{pmatrix}$ $\qquad X_2 = \begin{pmatrix} At^2 + Bt + C \\ Dt^2 + Et + F \end{pmatrix}$

or $X_p = \begin{pmatrix} Ae^{t} + Bt^2 + Ct + D \\ Ee^{t} + Ft^2 + Gt + H \end{pmatrix}$

replace in equation, solve for coefficients

the results should match our solution from above

8. $\vec{x}' = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}\vec{x}$

$(4-\lambda)(-1-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 - 6 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda = 5, -2$

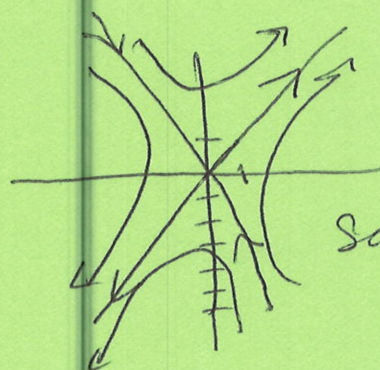
$y(t) = c_1e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$\lambda = 5$: $\begin{pmatrix} 4-5 & 1 \\ 6 & -1-5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}$ $\quad x_1 = x_2$

$\lambda = -2$: $\begin{pmatrix} 4+2 & 1 \\ 6 & -1+2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 6 & 1 \end{pmatrix}$ $\quad 6x_1 = -x_2$, $\quad x_1 = -\frac{1}{6}x_2$

$y_1 = e^{5t}$, $y_2 = e^{-2t}$

8. contd



Saddle point

9. $\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$ $\quad \frac{\partial N}{\partial x} = e^{xy} + xye^{xy}$

$\int 1 + ye^{xy}\,dx = x + e^{xy} + f(y)$ $\quad \varphi(x,y) = x + y^2 + e^{xy} + K$

$\int 2y + xe^{xy}\,dy = y^2 + e^{xy} + g(x)$

10. $xy' - 2y = x^3 \cos x$ $\quad \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$

$y' - \frac{2}{x}y = x^2 \cos x$

$x^{-2}y' - 2x^{-3}y = \cos x$ $\quad y(\pi) = 0$

$0 = \pi^2 \cancel{\sin\pi} + C\pi^2$

$\int (x^{-2}y)' = \int \cos x\,dx$ $\quad C = 0$

$x^{-2}y = \sin x + C$ $\quad y = x^2 \sin x$

$y = x^2 \sin x + Cx^2$

11. $y' + \frac{6}{x}y = 3y^{4/3}$ $\quad (1-n)y^{-n} = (1-\frac{4}{3})y^{-4/3} = -\frac{1}{3}y^{-4/3}$

$-\frac{1}{3}y^{-4/3}y' - \frac{2}{x}y^{-1/3} = -1$

12. a = ii $\quad$ b = iv $\quad$ c = i $\quad$ d = iii $\quad$ e = vi $\quad$ f = v

13. $y' + 2xy^2 = 0$ $\quad \int \frac{dy}{y^2} = \int -2x\,dx$ $\quad y = \frac{1}{x^2 + C}$

$y' = -2xy^2$

$-\frac{1}{y} = -x^2 + C$

14. a. nonlinear, first order

b. 4th order, linear

c. first order, nonlinear

d. fifth order, nonlinear

15. a. $r^2 - 2r + 2 = 0 \qquad r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$y(t) = c_1 e^t \cos t + c_2 e^t \sin t$

b. $2r^2 - r - 2 = 0 \qquad r = \frac{1 \pm \sqrt{1+16}}{2(2)} = \frac{1}{4} \pm \frac{\sqrt{17}}{4} = \frac{1 \pm \sqrt{17}}{4}$

$y(t) = C_1 e^{[(1+\sqrt{17})/4]t} + C_2 e^{[(1-\sqrt{17})/4]t}$

16. a. $A \sinh 3x + B \cosh 3x$

b. $Axe^x \cos x + Bxe^x \sin x$

c. $Axe^x + Bx^3 + Cx^2 + Dx + E$

17. a. $\mathcal{L}\{1 + 4t + 4t^2\} = \frac{1}{s} + \frac{4}{s^2} + \frac{8}{s^3}$

b. $\mathcal{L}\{e^{-2t} \sin 3t\} = \frac{3}{(s+2)^2 + 9}$

c. $\mathcal{L}\{\frac{1}{2}\int_0^t (t-\tau)^3 \sin 2\tau \, d\tau\} = \frac{1}{2}\mathcal{L}\{t^3\} \cdot \mathcal{L}\{\sin 2t\} = \frac{1}{2} \cdot \frac{6}{s^4} \cdot \frac{2}{s^2+4}$

$= \frac{6}{s^4(s^2+4)}$

d. $\mathcal{L}^{-1}\{\frac{1}{2} - \frac{2}{s^5}\} = \frac{1}{2}\delta(t) - \frac{1}{12}t^4$

e. $\mathcal{L}^{-1}\{\frac{9}{s^2+81} - \frac{17s}{s^2+81}\} = \sin 9t - 17 \cos 9t$

f. $\mathcal{L}^{-1}\{\frac{1}{(s-3)(s^2+4)(s^2-1)}\} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} + \frac{Ds+E}{s^2-1}$

$A(s^2+4)(s^2-1) + (Bs+C)(s-3)(s^2-1) + (Ds+E)(s-3)(s^2+4) =$

$As^4 + 3As^2 - 4A + Bs^4 - 3Bs^3 - Bs^2 + 3Bs + Cs^3 - 3Cs^2 - Cs + 3C + Ds^4 - 3Ds^3 + 4Ds^2$

$-12Ds + Es^3 - 3Es^2 + 4Es - 12E = 1$

17f contd.

$A + B + D = 0$ s^4

$3A - B - 3C + 4D - 3E = 0$ s^2

$-4A + 3C - 12E = 1$ 1

$-3B - 3D + E = 0$ s^3

$3B - C - 12D + 4E = 0$ s

$A = \frac{5}{196}$

$B = \frac{1}{245}$

$C = 3/49$

$D = -29/980$

$E = -15/196$

$$\mathcal{L}^{-1}\left\{ \frac{5/196}{s-3} + \frac{1/245\, s}{s^2+4} + \frac{3/49}{s^2+4} - \frac{+29/980\, s}{s^2-1} - \frac{15/196}{s^2-1} \right\}$$

$$= \frac{5}{196} e^{3t} + \frac{1}{245} \cos 2t + \frac{3}{98} \sin 2t - \frac{29}{980} \cosh t - \frac{15}{196} \sinh t$$

g. $\mathcal{L}^{-1}\left\{ \frac{e^{-\pi}}{s^2+1} \right\} = e^{-\pi} \sin t$

18. $y'' + 4y' - 12y = e^{-2t}$ $y(0) = 0,\ y'(0) = 1$

$$s^2 Y(s) - s(0) - 1 + 4sY(s) - 0 - 12Y(s) = \frac{1}{s+2}$$

$$Y(s)(s^2 + 4s - 12) = \frac{1 + s + 2}{s+2} = \frac{s+3}{s+2}$$

$$Y(s) = \frac{s+3}{(s+2)(s+6)(s-2)} = \frac{A}{s+2} + \frac{B}{s+6} + \frac{C}{s-2}$$

$$As^2 + 4As - 12A + Bs^2 - 4B + Cs^2 + 8Cs + 12C = s+3$$

$A + B + C = 0$

$4A + 8C = 1$

$-12A - 4B + 12C = 3$

$A = -\frac{1}{16}$

$B = -\frac{3}{32}$

$C = 5/32$

$$Y(s) = \frac{-\frac{1}{16}}{s+2} - \frac{3/32}{s+6} + \frac{5/32}{s-2}$$

$$y(t) = -\frac{1}{16} e^{-2t} - \frac{3}{32} e^{-6t} + 5/32\, e^{2t}$$

19. $y' = x + \sqrt[3]{y}$ $\quad y(0) = 1$ $\quad \Delta x = \frac{3-1}{10} = \frac{2}{10} = \frac{1}{5} = .2$

$n=0$ $\quad x_0 = 0$ $\quad y_0 = 1$ $\quad m_0 = (0 + \sqrt[3]{1})(.2) = .2$ $\quad y_1 = 1.2$

$n=1$ $\quad x_1 = .2$ $\quad y_1 = 1.2$ $\quad m_1 = (.2 + \sqrt[3]{1.2})(.2) = .25253..$ $\quad y_2 = 1.2 + .2525(.2)$

$x_2 = .4$ $\qquad = 1.2505$

$n=2$ $\quad y_2 = 1.2505$ $\quad m_2 = (.4 + \sqrt[3]{1.2505})(.2) = .29547...$

$\vdots$

$y_3 = 1.30959...$

20. $A(0) = 0$

$$\frac{dA}{dt} = \frac{5L}{min} \cdot \frac{40g}{L} - \frac{A}{1000-t} \cdot \frac{6L}{min}$$

$$\frac{dA}{dt} = 200 - \frac{6A}{1000t}$$

$$A' + \frac{6}{1000-t}A = 200 \qquad \mu = e^{\int \frac{6}{1000-t}} = e^{-6\ln(1000-t)} = (1000-t)^{-6}$$

$$(1000-t)^{-6}A' + 6(1000-t)^{-7}A = 200(1000-t)^{-6}$$

$$\left((1000-t)^{-6}A\right)' = \int 200(1000-t)^{-6}\,dt$$

$$(1000-t)^{-6}A = \frac{200}{5}(1000-t)^{-5} + C$$

$$A = 40(1000-t) + C(1000-t)^6$$ lasts for 1000 seconds

$$A(0) = 0 = 40(1000-0) + C(1000-0)^6$$

$$0 = 40,000 + C \cdot 10^{18}$$

$$\frac{-40,000}{10^{18}} = C \Rightarrow C = -4 \times 10^{-14}$$

$$A(t) = 40,000 - 40t - 4 \times 10^{-14}(1000-t)^6$$

when the tank drains, the amount will be 0 g.

8

21. $\vec{X} = \begin{pmatrix}4\\2\end{pmatrix}e^{2t} \quad X' = \begin{pmatrix}8\\4\end{pmatrix}e^{2t}$

$$\begin{pmatrix}3 & -2\\2 & -2\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}e^{2t} = \begin{pmatrix}12-4\\8-4\end{pmatrix}e^{2t} = \begin{pmatrix}8\\4\end{pmatrix}e^{2t}$$

22 a. $$\begin{vmatrix} e^t\sin t & e^t\cos t \\ e^t\sin t + e^t\cos t & +e^t\cos t - e^t\sin t \end{vmatrix} =$$

$$e^{2t}\sin t\cancel{\cos t} - e^{2t}\sin^2 t - e^{2t}\cancel{\sin t}\cos t - e^{2t}\cos^2 t$$

$$= -e^{-2t}(\sin^2 t + \cos^2 t) = -e^{2t}$$

b. $$\begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1$$

23. $(x-1)y'' - xy' + y = 0 \qquad y_1 = e^x \quad y_2 = e^x v$

$y_2' = e^x v + e^x v'$

$y_2'' = e^x v + 2e^x v' + e^x v''$

$$(x-1)(e^x v + 2e^x v' + e^x v'') - x(e^x v + e^x v') + e^x v = 0$$

$$e^x\left[\cancel{xv} + 2xv' + xv'' - \cancel{v} - 2v' - v'' - \cancel{xv} - xv' + \cancel{v}\right] = 0$$

$v''(x-1) + v'(x-2) = 0 \qquad u = v' \qquad \frac{du}{dx} = v''$

$u'(x-1) = -(x-2)u$

$$\int \frac{du}{u} = \int -\frac{(x-2)}{x-1}\,dx = \int -1 + \frac{1}{x-1}$$

$$\begin{array}{r} -1 \\ x-1\,\overline{)\,-x+2} \\ \underline{+x \mp 1} \\ 1 \end{array}$$

$\ln u = -x + \ln(x-1) = \ln e^{-x} + \ln(x-1) = \ln(e^{-x}(x-1))$

$u = (x-1)e^{-x} \qquad v = \int (x-1)e^{-x}\,dx = -xe^{-x}$

$y_2 = -xe^{-x} \cdot e^x = -x \qquad y(x) = c_1 e^x + c_2 x$

24. $y'' + 2y' + 5y = 0$ $\qquad r = \dfrac{-2 \pm \overset{\sqrt{16}\, i}{\sqrt{4-20}}}{2} = -1 \pm 2i$

$r^2 + 2r + 5 = 0$

$y_1 = e^{-t}\cos 2t \quad , \quad y_2 = e^{-t}\sin 2t$

a.

$Y(t) = A\sin 2t + B\cos 2t$

$Y'(t) = 2A\cos 2t - 2B\sin 2t$ $\qquad y(0) = 1, \ y'(0) = 3$

$Y''(t) = -4A\sin 2t - 4B\cos 2t$

$-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t + 5A\sin 2t + 5B\cos 2t = 3\sin 2t$

$\sin 2t \quad (-4A - 4B + 5A) = 3 \qquad A - 4B = 3 \qquad A = 3/17$

$\cos 2t \quad (-4B + 4A + 5B) = 0 \qquad 4A + B = 0 \qquad B = -12/17$

$y_p(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$

$y(0) = 1 = c_1(1)(1) + \cancel{c_2(1)(0)}^{0} + \cancel{\frac{3}{17}(0)}^{0} - \frac{12}{17}(1)$

$c_1 = \dfrac{29}{17}$

$y'(t) = -\frac{29}{17}e^{-t}\cos 2t - \frac{58}{17}e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t$

$+ \frac{6}{17}\cos 2t + \frac{24}{17}\sin 2t$

$y'(0) = 3 = \frac{-29}{17}(1)(1) - \cancel{\frac{58}{17}(1)(0)}^{0} - \cancel{c_2(1)(0)}^{0} + 2c_2(1)(1) + \frac{16}{17}(1) + \cancel{\frac{24}{17}(0)}^{0}$

$\dfrac{64}{17} = 2c_2 \qquad c_2 = \dfrac{32}{17}$

$y(t) = \frac{29}{17}e^{-t}\cos 2t + \frac{32}{17}e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$

b. $Y(t) = -e^{-t}\cos 2t \displaystyle\int \frac{3\sin 2t \cdot e^{-t}\sin 2t}{2e^{-2t}}\,dt + e^{-t}\sin 2t \int \frac{3\sin 2t\, e^{-t}\cos 2t}{2e^{-2t}}\,dt$

$= -\frac{3}{2}e^{-t}\cos 2t \int \sin^2 2t\, e^{t}\,dt + \frac{3}{2}e^{-t}\sin 2t \int \sin 2t\cos 2t \cdot e^{t}\,dt$

$= -\frac{3}{2}e^{-t}\cos 2t\left[\frac{-4}{17}e^{t}\sin 2t\cos 2t + \frac{1}{17}e^{t}\sin^2 2t + \frac{8}{17}e^{t}\right] +$

$\frac{3}{2}e^{-t}\sin 2t\left[-\frac{4}{34}e^{t}\cos 4t + \frac{1}{34}\sin 4t \cdot e^{t}\right] =$

24 cont'd

$\cos 2t \sin 2t \cdot \sin 2t$

$= \frac{6}{17}\cos^2 2t \sin 2t - \frac{3}{2}\cos 2t \sin^2 2t - \frac{12}{17}\cos 2t + -\frac{3}{17}\sin 2t \cos 4t$

$\cos 2t \cdot (\sin 2t \cos 2t)$ $\qquad + \frac{3}{68}\sin 4t \sin 2t$

$= \frac{6}{17}\cos^2 2t \sin 2t - \frac{3}{2}\cos 2t \sin^2 2t - \frac{12}{17}\cos 2t - \frac{3}{17}\sin 2t(\cos^2 2t - \sin^2 2t)$

$+ \frac{3}{68} 2\sin 2t \cos 2t \sin 2t$

$= \frac{6}{17}\cos^2 2t \sin 2t - \frac{3}{17}\sin 2t \cos^2 2t - \frac{3}{2}\cos 2t \sin^2 2t + \frac{3}{17}\sin 2t \sin^2 2t$

$+ \frac{3}{34}\sin^2 2t \cos 2t$

$= \underbrace{\frac{3}{17}\cos^2 2t \sin 2t + \frac{3}{17}\sin^2 2t \sin 2t} - \frac{12}{17}\cos 2t - \frac{24}{17}\cos 2t \sin^2 2t$?

$= \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$

solve the rest the way you did before

25. $24.3 = k(.3)$ $\qquad k = 81 \qquad m = 4$ $\qquad y(0) = -.2$

$y'(0) = 0$

$4y'' + 81y = 0$

$4r^2 + 81 = 0$

$r^2 = -\frac{81}{4} \qquad r = \pm \frac{9}{2}i$

$y(t) = c_1 \sin(\frac{9}{2}t) + c_2 \cos(\frac{9}{2}t)$

$y'(t) = \frac{9}{2}c_1 \cos(\frac{9}{2}t) - \frac{9}{2}c_2 \sin(\frac{9}{2}t)$

$y(0) = -.2 = \cancel{c_1(0)} + c_2(1)$

$c_2 = -.2$

$y'(0) = \frac{9}{2}c_1(1) - \cancel{\frac{9}{2}(-.2)}(0) = 0$

$c_1 = 0$

$y(t) = -.2\cos(\frac{9}{2}t)$

26. $T(0) = 425$

$T(2) = 350$

$T_R = 77$

$$\frac{dT}{dt} = k(T-77)$$

$$\frac{dT}{T-77} = k\,dt$$

$$\ln(T-77) = kt + C$$

$$T - 77 = T_i e^{kt} \Rightarrow T(t) = 77 + T_i e^{kt}$$

$$T(0) = 425 = 77 + T_i$$

$$T(t) = 77 + 348e^{kt}$$

$$T(2) = 77 + 348e^{k(2)} = 350$$

$$273 = 348e^{2k}$$

$$k = -.121365$$

$$T(\tau) = 120 = 77 + 348e^{-.12136t}$$

$t = 17.22899$ minutes